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Spectral decomposability of rank-one perturbations of normal operators

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ABSTRACT

This paper is a continuation of the study by Foias, Jung, Ko, and Pearcy (2007) [4] and Foias, Jung, Ko, and Pearcy (2008) [5] of rank-one perturbations of diagonalizable normal operators. In Foias, Jung, Ko, and Pearcy (2007) [4] we showed that there is a large class of such operators each of which has a nontrivial hyperinvariant subspace, and in Foias, Jung, Ko, and Pearcy (2008) [5] we proved that the commutant of each of these rank-one perturbations is abelian. In this paper we show that the operators considered in Foias, Jung, Ko, and Pearcy (2007) [4] have more structure – namely, that they are decomposable operators in the sense of Colojoară and Foias (1968) [1].

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all (bounded, linear) operators on $\mathcal{H}$. A natural question that arises in the theory of such operators is that of identifying classes of operators that have spectral properties akin to those of normal operators. (Recall that if N is a normal operator and $\mathcal{M}$ is any Borel subset of the spectrum of N, then there is a commuting idempotent $E(\mathcal{M})$ whose range and kernel are complementary hyperinvariant subspaces of N.) The best known class of operators in $\mathcal{L}(\mathcal{H})$ with spectral properties resembling those of normal operators is the class of spectral operators defined and developed by N. Dunford and his collaborators (cf., e.g., [3]). Subsequently other classes of operators with similar, albeit weaker, spectral properties were identified (see [3]). Among the latter, the class of decomposable operators is one of the largest. Therefore it is not surprising that, from time to time, some "concrete" operators have turned out to be decomposable (e.g., [6]). Below we show that this is also the case for the rank-one perturbations of diagonalizable normal operators studied by the authors in [4] and [5].

2. Preliminaries

The notation and terminology used herein agree almost completely with that of [4] and [5]. But for the readers' convenience, we repeat some of it. For T in $\mathcal{L}(\mathcal{H})$, we write $\{T\}'$ for the commutant of T and $\{T\}'' = (\{T\}')'$ for the double commutant of T. As usual in what follows, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{D}$ will denote the sets of positive integers, real numbers, complex numbers, and complex numbers of modulus less than one, respectively. We now choose an ordered orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ which will remain fixed throughout the paper. If $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ is any bounded sequence in $\mathbb{C}$, we write D_Λ for the normal operator in $\mathcal{L}(\mathcal{H})$ determined by the equations $D_\Lambda(e_n) = \lambda_n e_n$ for all $n \in \mathbb{N}$. This notation for $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ and D_Λ will also remain fixed throughout, as will the notation Λ' for the derived set of Λ. By definition, we shall say that

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an operator T in $\mathcal{L}(\mathcal{H})$ is a *rank-one perturbation of a diagonal normal operator* if there exist nonzero vectors $u = \sum_{n\in\mathbb{N}} \alpha_n e_n$ and $v = \sum_{n\in\mathbb{N}} \beta_n e_n$ in $\mathcal{H}$ and a bounded sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that T is unitarily equivalent to the operator $D_\Lambda + u \otimes v$, where, as usual, $u \otimes v$ is the operator of rank one defined by $(u \otimes v)(x) = \langle x, v\rangle u$, $x \in \mathcal{H}$. The notation $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ for the Fourier coefficients of u and v, respectively, will also remain fixed throughout this paper.

The two main results of [4] are as follows.

Theorem 2.1. *Let $T = D_\Lambda + u \otimes v$ be any rank-one perturbation of a diagonal normal operator such that $T \notin \mathbb{C}1_{\mathcal{H}}$ and $\sum_{n\in\mathbb{N}}(|\alpha_n|^{2/3} + |\beta_n|^{2/3}) < +\infty$. Then T has a nontrivial hyperinvariant subspace.*

To obtain this result we first dealt with some easy cases and then established the following.

Theorem 2.2. *With the notation as introduced above, suppose $T = D_\Lambda + u \otimes v$ is such that*

(i) *the map $n \to \lambda_n$ of $\mathbb{N}$ onto Λ is injective and Λ' is not a singleton,*
(ii) *for every $n \in \mathbb{N}$, $\alpha_n\beta_n \neq 0$, and*
(iii) $\sum_{n\in\mathbb{N}}(|\alpha_n|^{2/3} + |\beta_n|^{2/3}) < +\infty$ *(the nontrivial assumption).*

Then either

(I) *there exists an idempotent F with $0 \neq F \neq 1_{\mathcal{H}}$ such that $F \in \{T\}''$, and consequently, T has a complemented n.h.s.* (*i.e.*, *there exist n.h.s. $\mathcal{M}$ and $\mathcal{N}$ of T with $\mathcal{M} \cap \mathcal{N} = (0)$ and $\mathcal{M} + \mathcal{N} = \mathcal{H}$*), *or*
(II) *there exist an uncountable set $\{\mu\colon \mu \in P\}$ of eigenvalues of T and an associated family $\{u_\mu\}_{\mu\in P}$ of linearly independent eigenvectors (with $Tu_\mu = \mu u_\mu$) such that $\mathcal{M} = \bigvee_{\mu\in P}\{u_\mu\}$ is a n.h.s. for T and $\mathcal{H} \ominus \mathcal{M}$ is infinite dimensional.*

The ideal of compact operators in $\mathcal{L}(\mathcal{H})$ will be denoted by $\mathbf{K}$ and the Calkin map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbf{K}$ by π. For T in $\mathcal{L}(\mathcal{H})$ we denote by $\sigma(T)$ the spectrum of T, by $\sigma_{le}(T)$ [$\sigma_{re}(T)$] the left essential [right essential] spectrum of T, and

$$\sigma_e(T) = \sigma\big(\pi(T)\big) = \sigma_{le}(T) \cup \sigma_{re}(T), \qquad \sigma_{lre}(T) = \sigma_{le}(T) \cap \sigma_{re}(T).$$

Moreover, we write, as usual, $\sigma_p(T)$ for the point spectrum of T.

Definition 2.3. The class $(\mathcal{RO})$ will consist of all operators $T = D_\Lambda + u \otimes v$ in $\mathcal{L}(\mathcal{H})$ for which all coefficients α_n and β_n are nonzero, $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ is a one-to-one map of $\mathbb{N}$ into $\mathbb{C}$, and Λ' is not a singleton.

We remark that it follows easily that if $T_1 = D_{\Lambda_1} + u_1 \otimes v_1$ and $T_2 = D_{\Lambda_2} + u_2 \otimes v_2$ belong to $(\mathcal{RO})$ with $T_1 = T_2$, then the sequences Λ_1 and Λ_2 coincide and $u_1 \otimes v_1 = u_2 \otimes v_2$ (see [7, Prop. 1.1]). It is also clear that for all $T = D_\Lambda + u \otimes v \in (\mathcal{RO})$, we have $\sigma_e(T) = \sigma_{lre}(T) = \sigma_{lre}(D_\Lambda) = \Lambda'$.

One might expect that an arbitrary T in $(\mathcal{RO})$ would satisfy $\sigma_p(T) \cup \sigma_p(T^*) \neq \varnothing$ (and thus trivially have a n.h.s.), but that this is false has been known (in the case $D_\Lambda = D_\Lambda^*$) for at least fifty years (cf., e.g., [2]). Perhaps the first example of an operator $T \in (\mathcal{RO})$ such that Λ' has positive planar Lebesgue measure and $\sigma_p(T) = \varnothing$ was given by Stampfli [9].

Definition 2.4. Suppose $T = D_\Lambda + u \otimes v \in (\mathcal{RO}) \subset \mathcal{L}(\mathcal{H})$. If $\sigma(T) = \sigma_e(T)(= \Lambda')$, $\sigma(T)$ is a (perfect) connected subset of $\mathbb{C}$, and the sequences $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ satisfy

$$\sum_{n\in\mathbb{N}} |\alpha_n|^{2/3} < +\infty, \qquad \sum_{n\in\mathbb{N}} |\beta_n|^{2/3} < +\infty, \tag{2.1}$$

then T will be said to belong to the class $(\mathcal{RO})_1$.

Definition 2.5. For $T = D_\Lambda + u \otimes v$ in $(\mathcal{RO})_1$, we define $\gamma_n = \max\{|\alpha_n|, |\beta_n|\}$, $n \in \mathbb{N}$, and set

$$c_1^2 = \sum_{n\in\mathbb{N}} \gamma_n^{2/3} \quad (< +\infty). \tag{2.2}$$

Moreover, for $\zeta \in \mathbb{C}$ and $s > 0$, we define the open disc $\mathfrak{D}(\zeta, s)$ by

$$\mathfrak{D}(\zeta, s) := \big\{\lambda \in \mathbb{C}\colon |\lambda - \zeta| < s\big\},$$

and, in particular, we set, for every $r > 0$,

$$\Lambda_r := \bigcup_{n\in\mathbb{N}} \mathfrak{D}\big(\lambda_n, \gamma_n^{2/3} r\big), \qquad \Delta_r := \mathbb{C} \backslash \Lambda_r, \tag{2.3}$$

and $\Delta_0 := \bigcup_{r>0} \Delta_r$. Denoting planar Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$ by m_2, we obtain that

$$m_2(\Lambda_r) \leqslant \sum_{n\in\mathbb{N}} \pi \gamma_n^{4/3} r^2 = \pi r^2 \sum_{n\in\mathbb{N}} \gamma_n^{4/3}. \tag{2.4}$$

Remark 2.6. Note that if $T = D_\Lambda + u \otimes v \in (\mathcal{RO})_1$ and Λ consists – say – of the rational points in (the open unit disc) $\mathbb{D} = \mathfrak{D}(0,1)$ in $\mathbb{C}$, then $\sigma(T) = \sigma_{lre}(T) = \mathbb{D}^-$ and $m_2(\sigma(T)) = \pi$, so if r is chosen so small that $\pi r^2 \sum_{n\in\mathbb{N}} \gamma_n^{4/3}$ is very near 0, then Λ_r will still be an open covering of Λ, but the subset $\sigma(T) \cap \Delta_r$ will have m_2-measure almost π.

Remark 2.7. The underlying idea that enabled the basic constructions of [4] to be carried out is that even though $\sigma(T) \cap \Delta_r$ may be quite large, we were able to define the appropriate integrations over various simple closed Jordan curves that lie in Δ_r (for suitable $r > 0$), even though entire arcs on such curves may be contained in $\sigma(T)$.

Remark 2.8. The case in which $T \in (\mathcal{RO})_1$ and $\sigma_p(T) \cap \Delta_0$ is uncountable remains mysterious to the authors. We neither have an example of such a T nor can we prove that this phenomenon is impossible. The most we can say about such operators (if they exist) is that they satisfy conclusion (II) of Theorem 2.2.

We arrive finally at the class of operators that below will be shown to be decomposable.

Definition 2.9. We write $(\mathcal{RO})_2$ for the set of all $T = D_\Lambda + u \otimes v \in (\mathcal{RO})_1$ such that $\sigma_p(T) \cap \Delta_0$ is a countable set, $\|D_\Lambda\| < 1$, $\|T\| < 1$ (harmless normalizations), and

$$-1 < a := \min\{\mathrm{Re}(\lambda)\colon \lambda \in \sigma(T)\} < b := \max\{\mathrm{Re}(\lambda)\colon \lambda \in \sigma(T)\} < 1. \tag{2.5}$$

(This last normalization ($a < b$) is possible since $\sigma(T)$ is a perfect set and therefore has positive diameter, and we can replace T by $e^{i\theta}T$.)

Note that we may also suppose, without loss of generality, that all $r > 0$ under consideration satisfy $r \in (0, r_0)$, where

$$r_0 = \min\{1 - \|T\|, (b-a)/(4c_1^2)\}. \tag{2.6}$$

This ensures that for every $T \in (\mathcal{RO})_2$ and for $r \in (0, r_0)$ we have $\Lambda_r \subset \mathbb{D}$. It follows immediately from the definition of the set Λ_r in (2.3) and the connectedness of $\sigma(T) = \Lambda'$ that $P(\sigma(T)) = [a,b]$ and that $P(\Lambda_r)$ is a union of open subintervals of $\mathbb{R}$ of total length at most $2r \sum_{n\in\mathbb{N}} \gamma_n^{2/3} (= 2rc_1^2)$, where P is the projection of $\mathbb{C} = \mathbb{R}^2$ onto $\mathbb{R}$. Therefore

$$\Pi_r := (a,b) \backslash \big[P\big(\Lambda_r \cup \big(\sigma_p(T) \cap \Delta_0\big)\big)\big] \tag{2.7}$$

has (linear, Lebesgue) measure larger than $(b-a)(1 - rc_1^2) > (b-a)/2$ (since $\sigma_p(T) \cap \Delta_0$ is a countable (perhaps void) set). We note that an important and needed property of Π_r is that for every $s \in \Pi_r$, the vertical line $x = s$ lies entirely in Δ_r. We also will use the facts that the subset Π_r' consisting of all points of Π_r with Lebesgue density 1 has the same linear measure as does Π_r, and for each $s \in \Pi_r'$, there exist monotone sequences $\{s_n^-\}_{n\in\mathbb{N}}$ and $\{s_n^+\}_{n\in\mathbb{N}}$ in Π_r', with $a < s_n^- < s < s_n^+ < b$, such that $s_n^- \nearrow s$ and $s_n^+ \searrow s$. Moreover, observe that if $0 < r_1 < r_2 < r_0$, then $\Pi_{r_1} \supset \Pi_{r_2}$, so $\Pi_{r_1}' \supset \Pi_{r_2}'$.

3. Decomposability

Recall that an operator $A \in \mathcal{L}(\mathcal{H})$ has the *single-valued extension property* (SVEP) if for every connected open set $G \subset \mathbb{C}$ and every analytic function $w : G \to \mathcal{H}$ such that $(A - \lambda)w(\lambda) \equiv 0$ on G, one has $w \equiv 0$ on G.

Proposition 3.1. *Every $T \in (\mathcal{RO})_2$ has the* SVEP.

Proof. Let Ω be a connected open set in $\mathbb{C}$. Let $x(\lambda)$ ($\lambda \in \Omega$) be an $\mathcal{H}$-valued analytic function such that $(T - \lambda)x(\lambda) \equiv 0$ on Ω. If $\Omega \cap (\mathbb{C}\backslash\mathbb{D}) \neq \varnothing$, then trivially $x(\lambda) \equiv 0$ since $\sigma(T) \subset \mathbb{D}$. Thus we may suppose that $\Omega \subset \mathbb{D}$. Since $\mathrm{card}(\sigma_p(T) \cap \Delta_0) \leqslant \aleph_0$ and $m_2(\bigcap_{r>0} \Lambda_r) = 0$ (and therefore $m_2(\Delta_0 \cap \Omega) = m_2(\Omega)$), we must have $x(\lambda) = 0$ a.e. $[m_2]$ in Ω; consequently, by analyticity, $x(\lambda) \equiv 0$ in Ω. $\square$

We remind the reader that if an operator $A \in \mathcal{L}(\mathcal{H})$ has the SVEP and $x \in \mathcal{H}$, then one defines the *local resolvent* $\rho_A(x)$ of x with respect to A as the union of all open subsets of $\mathbb{C}$ for which there exists an $\mathcal{H}$-valued analytic function $x(\lambda)$ satisfying $(A - \lambda)x(\lambda) \equiv x$ on that open set, and the *local spectrum* of x with respect to A as $\sigma_A(x) := \mathbb{C}\backslash\rho_A(x)$. Obviously, $\rho_A(x) \supset \rho(A) := \mathbb{C}\backslash\sigma(A)$ for every $x \in \mathcal{H}$, and for every $S \subset \mathbb{C}$, one defines

$$\mathcal{H}_A(S) = \{x \in \mathcal{H}\colon \sigma_A(x) \subset S\}. \tag{3.1}$$

Recall next from [1] that a (closed) subspace $\mathcal{Y}$ of $\mathcal{H}$ is called a *spectral maximal space* of $A \in \mathcal{L}(\mathcal{H})$ if

(i) $A\mathcal{Y} \subset \mathcal{Y}$, and
(ii) for any subspace $\mathcal{L}$ of $\mathcal{H}$ with $A\mathcal{L} \subset \mathcal{L}$ and $\sigma(A|\mathcal{L}) \subset \sigma(A|\mathcal{Y})$, one has $\mathcal{L} \subset \mathcal{Y}$.

It is well known [1, Ch. 1, Prop. 3.2] that every spectral maximal space of A is hyperinvariant for A.

Finally, recall from [1] that an operator $A \in \mathcal{L}(\mathcal{H})$ is called *decomposable* if for every finite open covering $\{G_i\}_{1 \leqslant i \leqslant n}$ of $\sigma(A)$ there exists a collection $\{\mathcal{Y}_i\}_{1 \leqslant i \leqslant n}$ of spectral maximal spaces of A such that $\sigma(A|\mathcal{Y}_i) \subset G_i$, $1 \leqslant i \leqslant n$, and

$$\mathcal{H} = \mathcal{Y}_1 + \cdots + \mathcal{Y}_n.$$

The principal result of this note is the following.

Theorem 3.2. *Every operator T in $(\mathcal{RO})_2$ is decomposable.*

The proof of this theorem will be accomplished with the help of a sequence of propositions. Hereinafter, we shall suppose that the reader is familiar with the notation, terminology, and results from [4], but for his convenience, we review briefly [4, Th. 4.2]:

Theorem 3.3. *Let $T = D_\Lambda + u \otimes v \in (\mathcal{RO})_2$ and $r \in (0, r_0)$. Then for every $s \in \Pi_r'$, there exist two nonzero idempotents $F_j^s \in \{T\}''$, $j = 1, 2$, such that $F_1^s + F_2^s = 1_{\mathcal{H}}$ and $F_1^s \cdot F_2^s = 0$. Furthermore, for all $s, s' \in \Pi_r'$ with $s \neq s'$, and for $j = 1, 2$, $F_j^s \neq F_j^{s'}$.*

Sketch of the construction. Since $T \in (\mathcal{RO})_2$ and $r \in (0, r_0)$, we have $\sigma(T) \cup \sigma(D_\Lambda) = \Lambda' \cup \Lambda \subset \mathbb{D}$. We fix an arbitrary $s \in \Pi_r' \subset (a, b)$, so the vertical line segment $l_s \subset \mathbb{D}^-$ defined by

$$l_s = \left\{s + it\colon -\left(1 - s^2\right)^{1/2} \leqslant t \leqslant \left(1 - s^2\right)^{1/2}\right\}$$

lies entirely in $\Delta_r \cap \mathbb{D}^-$ and has endpoints on $\mathbb{T} := \partial\mathbb{D}$.

We next construct two positively oriented, piecewise smooth, simple closed, Jordan curves Γ_1^s, $\Gamma_2^s \subset \mathbb{T} \cup l_s$ as follows. Let Γ_j^s, $j = 1, 2$, consist of the line segment l_s together with an arc a_j^s of $\mathbb{T}$ (each properly oriented), where

$$a_1^s = \left\{e^{i\theta} \in \mathbb{T}\colon \operatorname{Re}\left(e^{i\theta}\right) \leqslant s\right\}, \qquad a_2^s = \left\{e^{i\theta} \in \mathbb{T}\colon s \leqslant \operatorname{Re}\left(e^{i\theta}\right)\right\}.$$

Note that both Γ_1^s and Γ_2^s contain l_s (with opposite orientations) as a subarc and are compact sets. Thus $\mathbb{T} = a_1^s \cup a_2^s \subset \rho(T) \cap \rho(D_\Lambda)$, so the resolvents $R_\lambda(T) = (\lambda - T)^{-1}$ and $R_\lambda(D_\Lambda)$ are analytic in a neighborhood of $\mathbb{T} = a_1^s \cup a_2^s$. It follows that for every $x \in \mathcal{L}$ (the dense linear manifold of [4, Th. 3.11]), the vector-valued integrals

$$F_j^s x := \frac{1}{2\pi i} \int\limits_{\Gamma_j^s} (\lambda - T)^{-1} x \, d\lambda \quad \left(= -\frac{1}{2\pi i} \int\limits_{\Gamma_j^s} x_\lambda^T \, d\lambda\right), \quad x \in \mathcal{L},\ j = 1, 2, \tag{3.2}$$

exist in the strong topology on $\mathcal{H}$. Moreover, since in the integral for $(F_1^s + F_2^s)x$, the integrations along l_s cancel one another, we get immediately that

$$\left(F_1^s + F_2^s\right)x = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} (\lambda - T)^{-1} x \, d\lambda, \quad x \in \mathcal{L},$$

and since $\sigma(T) \subset \mathbb{D}$, we see that also, by the Riesz–Dunford functional calculus, $F_1^s + F_2^s = 1_{\mathcal{H}}$. With some additional work (cf. [4]) we get that $F_1^s \cdot F_2^s = 0$.

Remark 3.4. As noted earlier, if $0 < r_1 < r_2 < r_0$, then $\Pi_{r_1}' \supset \Pi_{r_2}'$, so the idempotents F_1^s and F_2^s are actually independent of $r \in (0, r_0)$.

Remark 3.5. Note that by the Jordan curve theorem, each of the simple closed curves Γ_i^s, $i = 1, 2$, yields a separation of $\mathbb{C} \backslash \Gamma_i^s$ into two disjoint open regions, which we denote, as usual, by $\operatorname{Int}(\Gamma_i^s)$ and $\operatorname{Ext}(\Gamma_i^s)$.

Proposition 3.6. *For any $T \in (\mathcal{RO})_2$, $r \in (0, r_0)$, and $s \in \Pi_r'$, we have* $\operatorname{ran} F_j^s = \mathcal{H}_T(\mathbb{C} \backslash \operatorname{Ext} \Gamma_j^s)$, $j = 1, 2$.

Proof. Lemma 4.4 in [4] shows that

$$\operatorname{ran} F_j^s \subset \mathcal{H}_T\left(\mathbb{C} \backslash \operatorname{Ext} \Gamma_j^s\right), \quad j = 1, 2.$$

Let now $x \in \mathcal{H}$ such that $\sigma_T(x) \subset \mathbb{C} \setminus \operatorname{Ext} \Gamma_j^s$. We will treat only the case $j = 1$, since the other (i.e., $j = 2$) can be dealt with in a similar manner. So take $s' > s$, and note that

$$\sigma_T\bigl(F_2^{s'} x\bigr) \subset \sigma_T(x) \cap \sigma\bigl(T|_{\operatorname{ran} F_2^{s'}}\bigr) \subset \bigl(\mathbb{C} \setminus \operatorname{Ext} \Gamma_1^s\bigr) \cap \bigl(\mathbb{C} \setminus \operatorname{Ext} \Gamma_2^{s'}\bigr) = \varnothing,$$

so $F_2^{s'} x = 0$ (see [1, Ch. 1, Prop. 1.2(c)]). But as shown in the proof of [4, Step IV, pp. 641–642], $F_2^{s'} \to F_2^s$ (for $s' \searrow s$) in the weak operator topology (WOT), so that $F_2^{s'} x = 0$ for $s' > s$ gives $F_2^s x = 0$. This implies that

$$x = F_1^s x + F_2^s x = F_1^s x \in \operatorname{ran} F_1^s,$$

which completes the proof. □

By observing that $(\bigcup\{\Pi_r' \colon r \in (0, r_0)\})^- = [a, b]$, we readily infer the following.

Corollary 3.7. *For $T \in (\mathcal{RO})_2$ and for a dense set of s in $[a, b]$, the spectral maximal spaces*

$$\mathcal{H}_T\bigl(\{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda \leqslant s\}\bigr), \qquad \mathcal{H}_T\bigl(\{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda \geqslant s\}\bigr)$$

exist and are the ranges of the operators J_1^s and J_2^s in $\mathcal{L}(\mathcal{H})$, where

$$J_1^s + J_2^s = I, \qquad J_1^s = \bigl(J_1^s\bigr)^2, \qquad J_2^s = \bigl(J_2^s\bigr)^2, \qquad J_{1,2}^s \in \{T\}''$$

and

$$J_1^s J_1^{s'} = J_1^s, \qquad J_2^s J_2^{s'} = J_2^{s'} \quad \textit{for } s \leqslant s'.$$

4. Spectral idempotents

For convenience we make the following definition.

Definition 4.1. For $T \in (\mathcal{RO})_2$, an idempotent $J \in \mathcal{L}(\mathcal{H})$ will be called a *spectral idempotent* of T if:

(i) $J \in \{T\}''$,
(ii) $\operatorname{ran} J = \mathcal{H}_T(\sigma(T|\operatorname{ran} J))$.

Such a J will be denoted by $J_T(\sigma)$, where $\sigma = \sigma(T|\operatorname{ran} J)$.

According to Corollary 3.7, for $T \in (\mathcal{RO})_2$, $r \in (0, r_0)$, s in a dense subset of $[a, b]$, and $s' > s$,

$$J_1^s = J_T\bigl(\sigma(T) \cap \{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda \leqslant s\}\bigr), \tag{4.1}$$

and

$$J_2^{s'} = J_T\bigl(\sigma(T) \cap \bigl\{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda \geqslant s'\bigr\}\bigr). \tag{4.2}$$

Since for any two spectral maximal spaces $\mathcal{H}_T(\sigma_1)$, $\mathcal{H}_T(\sigma_2)$ of T, where $\sigma_j := \overline{\sigma}_j \subset \sigma(T)$, $j = 1, 2$, it is easy to see (from the definition) that the space

$$\mathcal{H}_T(\sigma_1) \cap \mathcal{H}_T(\sigma_2) = \mathcal{H}_T(\sigma_1 \cap \sigma_2)$$

is also a spectral maximal space of T, we have the following.

Lemma 4.2. *If $J_T(\sigma_1)$, $J_T(\sigma_2)$ (where $\sigma_j = \overline{\sigma}_j \subset \sigma(T)$, $j = 1, 2$) are two spectral idempotents of T, then $J_T(\sigma_1) J_T(\sigma_2)$ is the spectral idempotent $J_T(\sigma_1 \cap \sigma_2)$ of T.*

In order to facilitate the exposition we will extend the definition of the spectral idempotents to any $\sigma = \overline{\sigma} \subset \mathbb{C}$, as follows:

If $J_T(\sigma \cap \sigma(T))$ exists, then $J_T(\sigma) := J_T(\sigma \cap \sigma(T))$.

In this way we can reformulate the results obtained thus far as the following.

Lemma 4.3. *For $T \in (\mathcal{RO})_2$, let $s, s' \in \bigcup_{r\in(0,r_0)} \Pi'_r$ with $s \leqslant s'$. Then*

$$J_T\big(\{\lambda \in \mathbb{C}\colon\ s \leqslant \operatorname{Re}\lambda \leqslant s'\}\big) = J_1^{s'} J_2^{s},$$

where the J_1^s, $J_2^{s'}$ are as in (4.1), (4.2)*, respectively. In particular, $J_T(\{\lambda \in \mathbb{C}\colon\ \operatorname{Re}\lambda = s\}) = 0$.*

The aim of the next two propositions is to prove that the family of spectral idempotents of T contains a "sufficiently rich" Boolean algebra. To this end we note that instead of giving the real axis a privileged role we could have done all of our constructions by using the imaginary axis instead. Thus, if $[c, d]$ is the projection of $\sigma(T)$ on the imaginary axis (with $c < d$ by the normalization of T done in Definition 2.9), we obtain an $r''_0 > 0$ defined relative to $[c, d]$ analogously to (2.6) and the sets Π''_r analogous to the sets Π'_r, such that $\bigcup_{r\in(0,r''_0)} \Pi''_r$ is dense in $[c, d]$ and such that the analog of Lemma 4.3 is valid too. In particular, the following holds.

Lemma 4.4. *Let $s, s' \in \bigcup_{r\in(0,r_0)} \Pi'_r$, $t, t' \in \bigcup_{r\in(0,r''_0)} \Pi''_r$, $s \leqslant s'$, $t \leqslant t'$. Then the spectral idempotents*

$$J_T\big(\{\lambda \in \mathbb{C}\colon\ s \leqslant \operatorname{Re}\lambda \leqslant s'\}\big), \qquad J_T\big(\{\lambda \in \mathbb{C}\colon\ t \leqslant \operatorname{Im}\lambda \leqslant t'\}\big)$$

exist.

A rectangle $[s, s'] \times [t, t']$ in $\mathbb{C} = \mathbb{R}^2$ will be called *admissible* if $s, s' \in \bigcup_{r\in(0,r_0)} \Pi'_r$, $t, t' \in \bigcup_{r\in(0,r''_0)} \Pi'_r$ and $s \leqslant s'$, $t \leqslant t'$. Then

$$J_T\big(\big[s, s'\big] \times \big[t, t'\big]\big) = J_T\big(\{\lambda\colon\ s \leqslant \operatorname{Re}\lambda \leqslant s'\}\big) J_T\big(\{\lambda\colon\ t \leqslant \operatorname{Im}\lambda \leqslant t'\}\big)$$

exists. Moreover we set, by definition,

$$J_T\big([a, b] \times [c, d]\big) = I.$$

Now it is easy to check that the Boolean algebra $\mathcal{B}$ generated by the admissible rectangles (viewed as subsets of $[a, b] \times [c, d]$) is formed by all the finite unions of admissible rectangles $B = R_1 \cup R_2 \cup \cdots \cup R_n$ with mutually disjoint interiors. For a fixed such union we can now define

$$J_T(B) = J_T(R_1 \cup \cdots \cup R_n) = J_T(R_1) + J_T(R_2) + \cdots + J_T(R_n).$$

Since $J_T(R_i) J_T(R_j) = 0$ if $i \neq j$, the definition is consistent and

$$\operatorname{ran} J_T(R_1 \cup \cdots \cup R_n) = \mathcal{H}_T(R_1 \cup \cdots \cup R_n) = \mathcal{H}_T(B).$$

Thus this definition does not depend on the particular representation of B as a union of admissible rectangles. It is easy (although somewhat tedious and a familiar argument from measure theory) to see that

$$B(\in \mathcal{B}) \mapsto J_T(B) \in \mathcal{L}(\mathcal{H}) \tag{4.3}$$

is finitely additive, so summing up, we obtain the following.

Proposition 4.5. *The map* (4.3) *is a finitely additive* "*measure*" *on the Boolean algebra $\mathcal{B}$ generated by the admissible rectangles with values in the family of spectral idempotents of T.*

Finally, we are in a position to prove Theorem 3.2.

Proof of Theorem 3.2. Let $G_1, \ldots, G_N$ be open subsets of $\mathbb{C}$ such that

$$\sigma(T) \subset G_1 \cup G_2 \cup \cdots \cup G_N.$$

Obviously there exist open sets $\mathcal{O}_1, \ldots, \mathcal{O}_N$ such that

$$\sigma(T) \subset \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_N,$$

and $\overline{\mathcal{O}}_j \subset G_j$, $1 \leqslant j \leqslant N$. Let

$$\delta = \min_j \big\{\operatorname{dist}(\overline{\mathcal{O}}_j, \mathbb{C}\backslash G_j)\big\} = \min_j \big\{\min\big\{|\lambda - \mu|\colon\ \lambda \in \overline{\mathcal{O}}_j,\ \mu \notin G_j\big\}\big\},$$

which is obviously positive.

The construction now proceeds as in the construction of an ordinary measure from a more primitive set function defined on finite unions of admissible rectangles. We place a grid on the rectangle $[a,b]\times[c,d]$ by introducing intermediate points

$$a = x_1 < x_2 < \cdots < x_m = b, \qquad c = y_1 < y_2 < \cdots < y_n = d, \tag{4.4}$$

in a such a way that the mesh of each partition in (4.4) is less than $\delta/4$, and choose points $s_1,\dots,s_m \in \bigcup_{r\in(0,r_0)} \Pi'_r$ and $t_1,\dots,t_n \in \bigcup_{r\in(0,r''_0)} \Pi''_r$ such that

$$a = x_1 = s_1 < x_2 < s_2 < \cdots < s_{m-1} < x_m = s_m = b, \qquad c = y_1 = t_1 < y_2 < t_2 < \cdots < t_{n-1} < y_n = t_n = d.$$

Then $s_{i+1} - s_i < \delta/2$, $t_{j+1} - t_j < \delta/2$ ($i = 1,\dots,m-1$; $j = 1,\dots,n-1$), so any (admissible) rectangle $[s_i,s_{i+1}]\times[t_j,t_{j+1}]$ which intersects some $\overline{\mathcal{O}}_k$ will be included in G_k. Let

$$\sigma_k = \bigcup\bigl([s_i,s_{i+1}]\times[t_j,t_{j+1}]\bigr),$$

where the union is taken over all admissible rectangles which intersect $\overline{\mathcal{O}}_k$ ($k = 1,2,\dots,N$). Then

$$\mathcal{H}_T(\sigma_k) = \operatorname{ran} J(\sigma_k), \quad \sigma_k \subset G_k \quad (k = 1,2,\dots,N). \tag{4.5}$$

Moreover,

$$I = \sum_{1\leqslant i\leqslant m-1,\,1\leqslant j\leqslant n-1} J_T\bigl([s_i,s_{i+1}]\times[t_j,t_{j+1}]\bigr) = \sum_{([s_i,s_{i+1}]\times[t_j,t_{j+1}])\cap\sigma(T)\neq\varnothing} J_T\bigl([s_i,s_{i+1}]\times[t_j,t_{j+1}]\bigr);$$

hence

$$\mathcal{H} = \sum_{([s_i,s_{i+1}]\times[t_j,t_{j+1}])\cap\sigma(T)\neq\varnothing} \operatorname{ran} J_T\bigl([s_i,s_{i+1}]\times[t_j,t_{j+1}]\bigr).$$

But any rectangle $[s_i,s_{i+1}]\times[t_j,t_{j+1}]$ which intersects $\sigma(T)$ is contained in some σ_k, and therefore the corresponding

$$\operatorname{ran} J_T\bigl([s_i,s_{i+1}]\times[t_j,t_{j+1}]\bigr) = \mathcal{H}_T\bigl([s_i,s_{i+1}]\times[t_j,t_{j+1}]\bigr)$$

is included in $\mathcal{H}_T(\sigma_k)$. It follows that

$$\mathcal{H} \subset \sum_{k=1}^{N} \mathcal{H}_T(\sigma_k) \subset \mathcal{H}, \quad \text{i.e.,} \quad \mathcal{H} = \sum_{k=1}^{N} \mathcal{H}_T(\sigma_k).$$

This with (4.5) shows that, indeed, T is a decomposable operator. □

Remark 4.6. We remark again (cf. Remarks 2.6 and 2.7) that the operators $T = D_\Lambda + u\otimes v$ in $(\mathcal{RO})_2$, shown to be decomposable by Theorem 3.2, may have any perfect compact set in $\mathbb{C}$ as spectrum. In particular, $m_2(\sigma(T))$ may be any positive number. Of course, in the special case in which Λ lies on $\mathbb{R}$ or $\mathbb{T}$ and the (much more general) perturbation of D_Λ simply lies in the Macaev ideal (cf. [8]), the fact that T is decomposable has been known for several decades (cf. [1]).

5. Open questions

Our work on this topic – [4,5], together with the present note, answers some questions about rank-one perturbations of diagonalizable normal operators, but leaves many problems unresolved. Here are some of them.

Problem 5.1. Does every rank-one perturbation of a diagonalizable normal operator have a nontrivial invariant (or hyperinvariant) subspace?

Problem 5.2. Does there exist an operator in $(\mathcal{RO})_2$ with uncountable point spectrum?

Problem 5.3. What can be said about the structure of rank-one perturbations of non-diagonalizable normal operators (perhaps of multiplicity one)?

Problem 5.4. Does there exist an analog of the constructions in [4] and herein that could be applied to a more general class of finite rank perturbations of diagonalizable normal operators? It seems to the authors that the condition placed on the Fourier coefficients of u and v for $T = D_\Lambda + u\otimes v$ in $(\mathcal{RO})_2$ is a sort of "regularity requirement" that, properly understood, might be applicable to a certain class of operators of the form $D_\Lambda + \sum_{i=1}^{N} u_i\otimes v_i$ to yield comparable results.

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